Controlling the multistability of nonlinear systems with coexisting attractors

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(Received 16 March 2001; published 19 September 2001)

A method for controlling generalized multistability is suggested. The method implies the application of a small harmonic modulation with properly chosen frequency and amplitude to a system parameter. The possibility of control is demonstrated with the example of the Hénon map with coexisting period-1, period-3, and period-9 attractors. It is shown that one or more coexisting attractors can lose their stability and the attractor can undergo crisis when the control frequency is close to the relaxation oscillation frequency or its subharmonics of the corresponding attractor.

DOI: 10.1103/PhysRevE.64.046203 PACS number(s): 05.45.Gg, 05.45.Ac

The coexistence of multiple attractors is one of the most exciting phenomena in nonlinear dynamics. This phenomenon, referred to as generalized multistability [1], has been observed in various systems, including electronic circuits [2], lasers [3], and mechanical [4] and biological systems [5]. However, very often the coexistence of multiple attractors is not desirable. For instance, in a laser with intracavity frequency doubling the multistability can result in instabilities in the laser intensity, known as the green problem [6]. Notwithstanding the important problem of controlling multistable systems, there are no efficient methods for such control to our knowledge. It seems that a proper change in initial conditions might be appropriate in this situation, for example, in the form of a short external impact [7]. However, in many cases it is not possible to switch the system on and off, for instance, for some kind of biological or chemical processes. Moreover, there is no guarantee that the system after such a switch will change its state or that some noise or instability will force the system to jump back from the selected state.

It is also possible to provide some regulation of the steady state of a nonlinear system through adaptive control mechanisms [8], which utilize an error signal proportional to the difference between the goal output and the actual trajectory. This error signal governs the change of the parameters of the system so as to reduce the error to zero. This is one of the feedback control methods, which require an appropriate feedback loop and permanent tracking of the system state in order to apply the control as soon as the system switches to another coexisting attractor due to, e.g., noise or any other instability. Another control algorithm to drive trajectories to a desirable attractor by using small feedback control has been suggested by Lai [9]. His idea is to build a bushlike structure of paths to the target attractor and to stabilize a trajectory around one of the many paths so that the trajectory asymptotes to the desirable attractor. However, at present there is no guarantee that the method can be used in practical applications [10]. The success of the method relies on the region in the phase space to which the bush extends and the method is effective when there are fractal basin boundaries with large values of fractal dimension. In contrast, there is no appreciable increase in the probability for a trajectory to be driven to the desirable attractor if the basin boundaries are smooth. We also remark that the feedback control methods require a prior knowledge of the system behavior and therefore they are not appealing for systems whose state is impossible or difficult to measure in real time and where a feedback loop is hard to realize, for example, for some medical and biological applications.

There exists another approach for controlling a system with coexisting attractors, namely, annihilation of an undesirable attractor. It was shown in a recent Letter [11] that a parameter modulation with properly chosen frequency and amplitude can destroy one of the coexisting attractors in a bistable system. Such a modulation can cause a crisis in the attractor that leads to its destruction. However, a new question arises: Is it possible to affect the attractors selectively in order to annihilate simultaneously various attractors in a multistable system for the purpose of making the system monostable? This work provides a positive answer to this question. Here we will demonstrate with the example of the multistable Hénon map how multiple attractors can be destroyed by nonfeedback parameter modulation, and will show how optimal conditions can be chosen for realization of this method.

The Hénon map is one of the simplest systems that display generalized multistability. The Hénon map [12] is described by

$$x_{n+1} = 1 - \mu x_n^2 + y_n, \tag{1}$$

$$y_{n+1} = -J x_n, \tag{2}$$

where x_n and y_n can be measured as time series, $J=0.9$ is the Jacobian related to dissipation, and μ is the control parameter. We found that the map Eqs. (1) and (2) exhibits the coexistence of three attractors (period 1, period 3, and period 9) in the parameter range $1.077<\mu<1.089$. The period-3 attractor coexists with the period-1 attractor in the range $0.92<\mu<1.18$. The bifurcation diagram for the uncontrolled case in shown in Fig. 1. The attractors have different relaxation oscillation frequencies f_r, which depend on the parameter μ. The frequency f_r can be measured numerically from a time series as the frequency of damped oscillations after a small disturbance from the equilibrium point. The depen-

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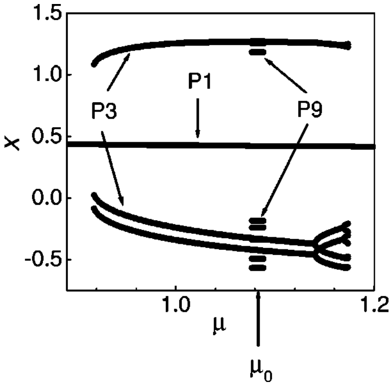

FIG. 1. Bifurcation diagram of the Hénon map with coexisting period-1, period-3, and period-9 attractors. The arrow indicates the initial position of the parameter $\mu_0 = 1.083$ where the control is applied.

dences of f_r on μ for each attractor are shown in Fig. 2. For example, at $\mu = 1.089$ the relaxation oscillation frequencies for the period-3 and period-9 attractors are $f_r^{(3)} = 0.106$ and $f_r^{(9)} = 0.0275$. In fact, f_r is equal to the imaginary part of the eigenvalue of the corresponding fixed point. However, it is not easy to find these values analytically because even for the period-3 attractor one needs to solve an eighth-order characteristic algebraic equation.

Since each attractor has its own characteristic frequency, it becomes possible to act selectively on the desired attractor by modulating a system parameter with a properly chosen frequency. The control is applied in the form of the harmonic modulation

$$\mu = \mu_0 + \mu_c \sin(2\pi f_c n), \tag{3}$$

where μ_c and f_c are the amplitude and frequency of the control. The initial value of the control parameter, $\mu_0 = 1.083$, is fixed at the middle point of the period-9 branch (shown by the arrow in Fig. 1). The control amplitude μ_c is considered to be relatively small, so that no qualitative changes in the system behavior occur for the uncontrolled case, if μ is changed from μ_0 to $\mu_0 \pm \mu_c$. Also, the modulation is a slow one in the sense that $f_c \ll 1$. Nevertheless, as we will show below, such a weak and slow modulation with a properly chosen f_c enables one to drastically affect the organization of the attractors and to annihilate one or more coexisting attractors.

In Fig. 3 we plot the annihilation curves (stability boundaries) for the period-9 and period-3 attractors in the space of the control parameters μ_c and f_c. When the control amplitude is above the annihilation curve, the corresponding state is destroyed. The horizontal lines indicate the level of the modulation amplitude at which the control parameter crosses the boundaries of attraction in the quasistationary case, i.e., when $f_c \to 0$. As seen from the figure, the annihilation curves have several extrema. The optimal conditions for attractor annihilation (the minimal control amplitude) are realized when f_c is close to f_r of the corresponding attractor. The other extremum in Fig. 3(a) appears at a subharmonic frequency, i.e., when $f_c = f_r^{(3)}/2$.

The effect of the control modulation on the destruction of the coexisting attractors is demonstrated with the time series shown in Fig. 4. In this figure we illustrate the consecutive annihilation of the coexisting attractors. Initially, prior to the control ($n < n_1$) three periodic attractors coexist at $\mu_0 = 1.083$. Starting from the initial conditions for period 9, we apply the control modulation Eq. (3) with $\mu_c = 0.0029$ and $f_c = 0.0275$ at $n = n_1$. The parameters of the control modulation are chosen to lie above the annihilation curve shown in Fig. 3(b), but to be close to the curve. Since the annihilation curves in Fig. 3 are stability boundaries of the corresponding attractors, the transient time increases when the parameters approach the curves. After the transients the period-9 attractor undergoes crisis that leads to the attractor destruction, and the system goes to period 3. A small-amplitude response to the control modulation is seen in the period-3 state. However, the control frequency is too far from the resonance frequency for the period-3 attractor and the control amplitude is too low to have an effect on the period-3 state. In order to make the system monostable, we change the param-

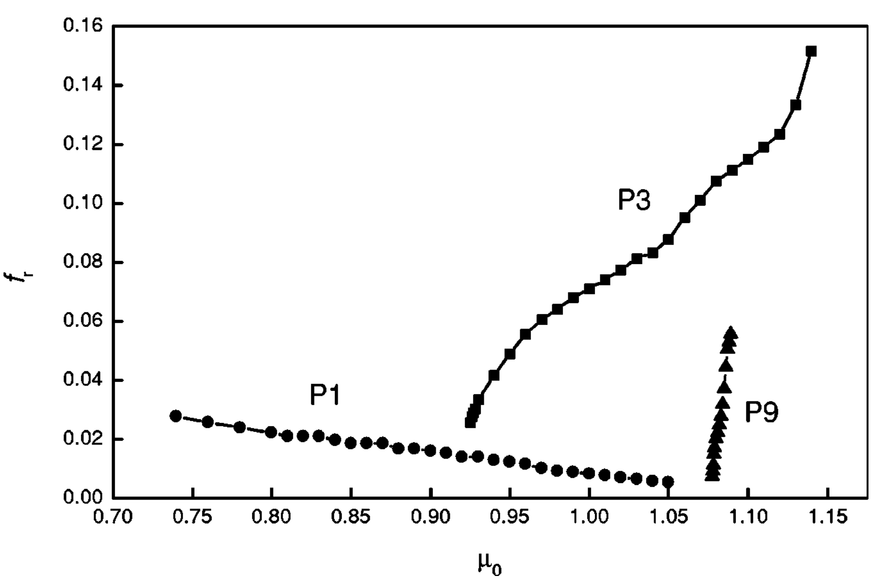

FIG. 2. Relaxation oscillation frequencies of the period-1 (dots), period-3 (squares), and period-9 (triangles) attractors versus control parameter.

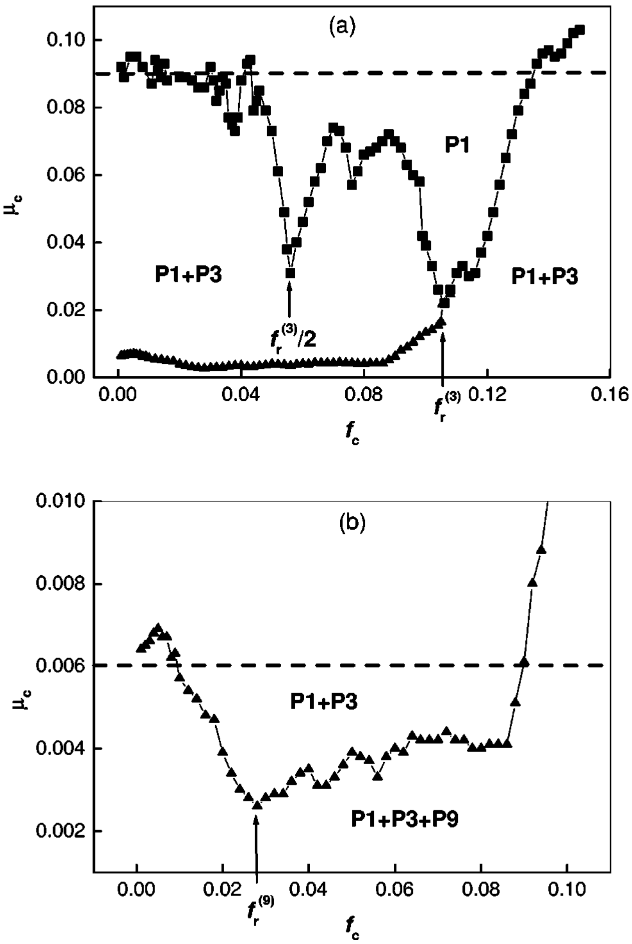

FIG. 3. Annihilation curves for the period-3 (squares) and period-9 (triangles) attractors in the space of the modulation frequency and amplitude for $\mu_0 = 1.083$. The optimal conditions for annihilation of period 3 and period 9 are realized, respectively, when (a) $f_c = f_r^{(3)} = 0.106$ or $f_c = f_r^{(3)}/2 = 0.053$ and (b) $f_c = f_r^{(9)} = 0.0275$ (shown by the arrows). The corresponding attractor is destroyed above the annihilation curve, while below these curves the three attractors coexist. The dashed lines indicate the maximal control amplitude at which the system changes the attractor when $f_c \to 0$.

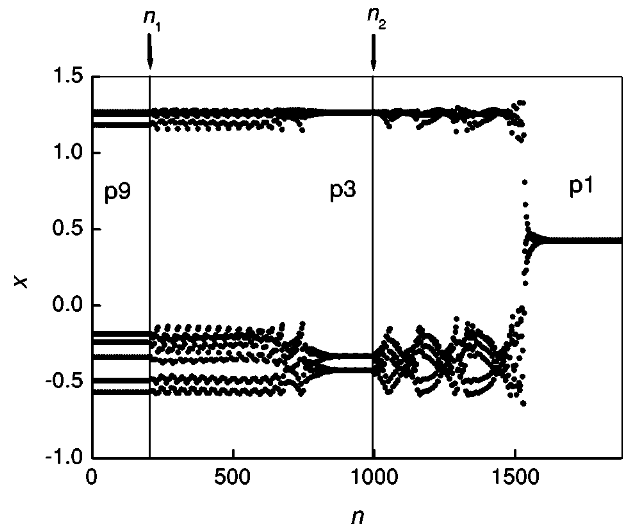

FIG. 4. Temporal evolution of the Hénon map with coexisting attractors under the control modulation. Prior to the control ($n < n_1$), three attractors coexist: the period 9 (p9), period 3 (p3), and period 1 (p1). The control parameters $\mu_c = 0.0029$ and $f_c = 0.0275$ at $n \geqslant n_1$, and $\mu_c = 0.023$ and $f_c = 0.106$ at $n \geqslant n_2$. The arrows show the instant of time when the control parameters are changed. The period-9 and period-3 attractors are destroyed after transients.

eters of the control modulation at the instant of time $n = n_2$ to $\mu_c = 0.023$ and $f_c = 0.106$. These parameters lie above the annihilation curve for the period-3 attractor [Fig. 3(a)], and hence this control leads to crisis of the period-3 attractor (after transients) followed by a system switch to the remaining period-1 state. The periodic modulation is hardly seen in the system response in the period-1 attractor, because it has no resonance frequency (the solution is real) for this value of μ_0 (see Fig. 2).

The physical mechanism of the annihilation phenomenon may be examined by analyzing the basins of attraction shown in Fig. 5. In this figure we illustrate in a phase space plot the situation shown in Fig. 4. The dots inside the basins have a different color for each attractor: yellow for the attractor of period 1, blue for the period-3 attractor, and gray for the period 9. The parameter modulation creates periodic orbits in the vicinity of the fixed points [red lines in Fig. 5(b)]. One can see that the amplitude of the system response is relatively small when f_c is very different from f_r and the periodic orbits are located inside the corresponding basins of attraction. However, when f_c approaches the resonance frequency f_r or its subharmonics, the control modulation is amplified by the system and the trajectory (black points in Fig. 5) becomes more extensive and crosses the basin boundaries. The oscillations may induce period doubling and chaos in the control modulation and finally cause the attractor to undergo crisis (see Fig. 4). As soon as the trajectory hits the basin of another attractor, it becomes attracted by the other attractor and approaches the other state along the arrows shown in Fig. 5. It should be noted that in the final state the system response is modulated with the frequency f_c, but the modulation amplitude is very low because f_r of the target attractor is very different from f_c. The modulation amplitude also plays a significant role in the attractor annihilation. Although μ_c is small, it should be large enough in order that the trajectory for the resonant frequency crosses the basin boundary.

Of course, the diagram shown in Fig. 5 is a very rough illustration of the dynamical processes leading to attractor annihilation. Recall that the basins shown in Fig. 5 belong to the uncontrolled system. However, strictly speaking, the parameter modulation not only creates periodic orbits, it also changes the organization of the basins of attraction [13]. In other words, the new attractors created by the parameter modulation have different basins of attraction, yet this change in the attractor boundaries has no effect on the general interpretation of the phenomenon. The results on the deformation of attractor boundaries in the modulated Hénon map will be published elsewhere [14].

To conclude, in this paper we have extended the method of attractor annihilation introduced in Ref. [11] to dynamical

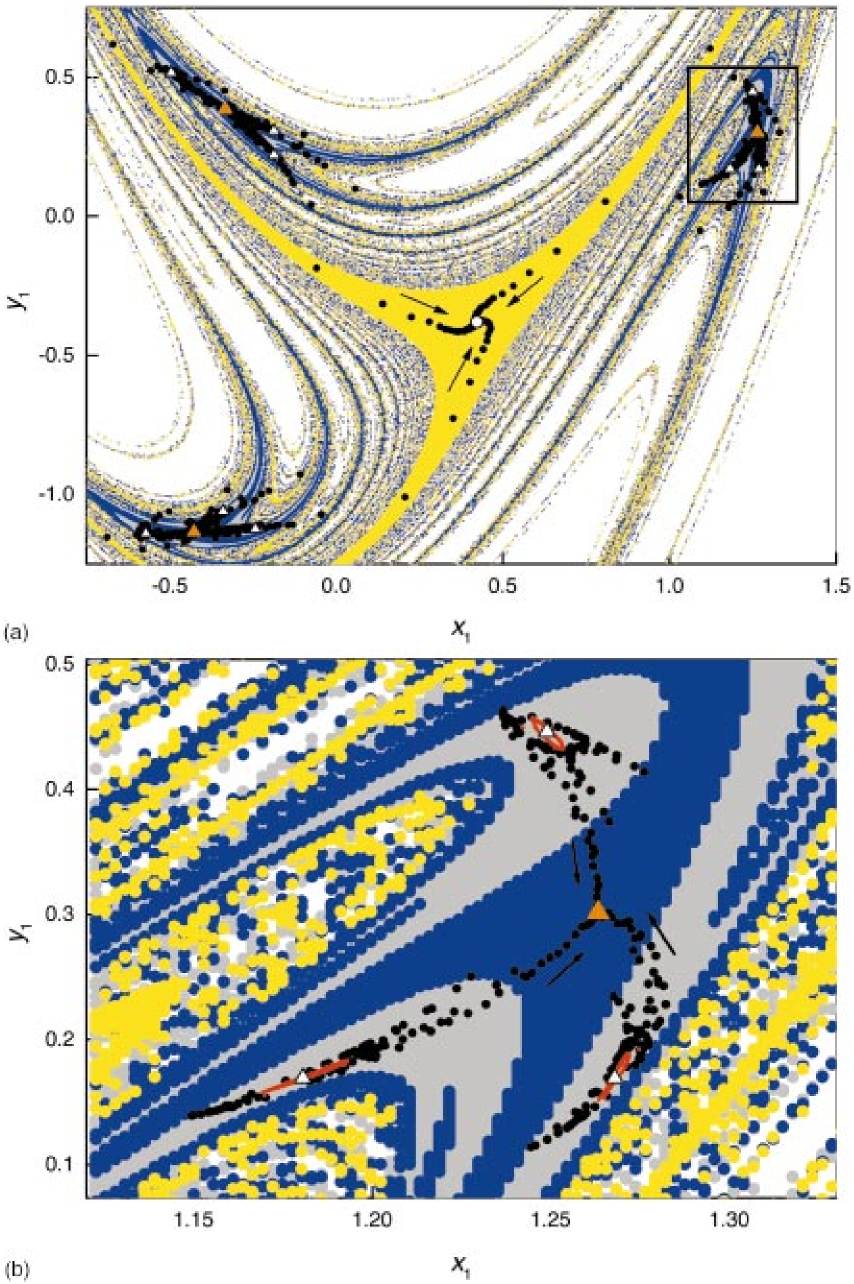

FIG. 5. (Color) Basins of attraction of the period-1 (yellow dots), period-3 (blue dots), and period-9 (gray dots) attractors. (a) Annihilation of the period-3 attractor at $\mu_c = 0.023$ and $f_c = 0.106$. (b) Enlarged diagram of the rectangular box in (a) showing the annihilation of the period-9 attractor at $\mu_c = 0.0029$ and $f_c = 0.0275$. The period-1, period-3, and period-9 attractors are shown, respectively, by the white dot, red triangles, and white triangles. The new period-9 orbits created by the modulation with $\mu_c = 0.0029$ and $f_c = 0.01$ are shown by the red lines. The arrows indicate the directions of the trajectories (black dots) created by the resonant modulation leading to the attractor annihilation.

systems with multiple coexisting attractors resulting in the selective control of generalized multistability. With the example of the Hénon map, we have demonstrated that a small harmonic parameter modulation can annihilate multiple attractors and make the system monostable. The optimal conditions for control are achieved when the control frequency is close to the frequency of relaxation oscillations of the attractor to be annihilated or its subharmonics. A similar approach was recently applied to a continuous system, namely, to the Duffing equations with three coexisting attractors [15] and to a CO_2 laser with delayed feedback [16]. The results were very similar and demonstrate the universality of the method. An important point is the applicability of the method to other dynamical systems. As seen from Fig. 2 the method can be applied to systems that undergo damped oscillations to an equilibrium point so that f_r is finite and nonzero. Mathematically, this condition requires complex eigenvalues for a fixed point or a periodic orbit, i.e., the fixed point should be a focus. We believe the results of this work may have many applications, for example, for stabilization of a solid state laser with intracavity second harmonic generation or a semiconductor laser with optical injection where generalized multistability has recently been detected [17]. The latter laser is often used in communications and the control of multistability is of great interest for such a system.

This work was supported in part by Consejo Nacional de Ciencia y Tecnología de México (CONACYT) (Project No. 33769-E) and through a grant from the Institute Mexico-USA of the University of California (UC MEXUS) and CONACYT. The author also thanks J. M. Saucedo for his assistance in the numerical simulations.

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